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Axially Symmetric Brans–Dicke–Maxwell Solutions

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Following a method of Johri and Goswami new solutions of coupled Brans–Dicke–Maxwell theory are generated from Zipoy's solutions in oblate and prolate spheroidal coordinates for source-free gravitational field. All these solutions become Euclidean at infinity. The asymptotic behavior and the singularity of the solutions are discussed and a comparative study made with the corresponding Einstein–Maxwell solutions. The possibility of a very large red shift from the boundary of the spheroids is also discussed.

1. INTRODUCTION

With the discovery of pulsars, which are dense, slowly rotating stars with large magnetic fields, there has been a resurgence of interest in the solutions of Einstein–Maxwell and Brans–Dicke–Maxwell field equations in the spheroidal coordinates, primarily because the equilibrium shape of the pulsars is supposed to be oblate spheroid and they are so far the only astronomical objects where general relativistic effects are not negligible.

It is well known that Einstein's equations are not completely Machian in nature. To make things more consistent with Mach's principle and less reliant on absolute properties of space, Brans and Dicke (1961) introduced the concept of a scalar field. A number of authors have discussed methods of generating new solutions of these equations from known solutions of Einstein's equations. Among them are included Geroch (1971, 1972), Buchdahl (1972), McIntosh (1974), and S. Chatterjee and S. Banerji (1980). In a recent paper Johri and Goswami (1979) have developed a method for generating new solutions of B–D–Maxwell equations from known solutions of Einstein's field equations. In this paper we have applied their method to get new B–D–Maxwell solutions, using Zipoy's solutions as source equations. The plan of our paper is as following: We summarize Zipoy's results

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in Section 2. In Section 3 we give in brief Johri and Goswami's method. In Sections (4) and (5) solutions in oblate and prolate coordinates are discussed. The paper ends with a brief discussion on the possible large red shift in our metric.

2. ZIPOY'S SOLUTIONS

In Weyl canonical form the axially symmetric static line element may be written in cylindrical coordinates as

$$ds^2 = e^{2\sigma} dt^2 - e^{2(\lambda-\sigma)}(d\rho^2 + dz^2) - \rho^2 e^{-2\sigma} d\varphi^2 \tag{1}$$

where $\sigma = \sigma(\rho, z)$, $\lambda = \lambda(\rho, z)$. Oblate spheroidal coordinates (u, θ) are connected with (ρ, z) through

$$\rho = a \cosh u \cos\theta, \qquad z = a \sinh u \sin\theta \tag{2}$$

where $0 \leqslant u \leqslant \infty$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \varphi \leqslant 2\pi$, $-\infty < t < \infty$. The line element then reduces to the form

$$ds^2 = e^{2\sigma} dt^2 - a^2 e^{2(\lambda-\sigma)}(\sin h^2 u + \sin^2\theta)(du^2 + d\theta^2)$$

$$- a^2 e^{-2\sigma} \cosh^2 u \cos^2\theta \, d\varphi^2 \tag{3}$$

Zipoy (1966) showed that the solutions for the "Newtonian potential" σ can be written as a linear combination of Legendre polynomials of integral order L.

For $L=0$, he got

$$\sigma = -\beta \tan^{-1} \operatorname{csch} u \tag{4}$$

$$\lambda = \frac{1}{2}\beta^2 \ln \frac{\sinh^2 u + \sin^2\theta}{\cosh^2 u} \tag{5}$$

where $\beta = m/a$ (m may be identified with the mass of the source a for asymptotic flatness).

In prolate spheroided coordinates

$$ds^2 = e^{2\sigma} dt^2 - a^2 e^{2(\lambda-\sigma)}(\sinh^2 u + \cos^2\theta)(du^2 + d\theta^2)$$

$$- a^2 e^{-2\sigma} \sinh^2 u \cos^2\theta \, d\varphi^2 \tag{6}$$

where

$$\rho = a \sinh u \cos\theta, \qquad z = a \cosh u \sin\theta \tag{7}$$

For $L=0$

$$\sigma = -\frac{\beta}{2} \ln \frac{\cosh u + 1}{\cosh u - 1} \tag{8}$$

$$\lambda = -\frac{\beta^2}{2} \ln \frac{\sinh^2 u + \cos^2\theta}{\sinh^2 u} \tag{9}$$

3. JOHRI AND GOSWAMI'S METHOD

For source-free electromagnetic fields the Brans–Dicke equations reduce to

$$R_{ij} = -\frac{8\pi}{\phi} T_{ij} - \frac{\omega}{\phi^2} \phi_{,i}\phi_{,j} - \frac{\phi_{i;j}}{\phi} \tag{10}$$

$$\phi^{k}_{;k} = 0 \qquad (\omega \neq -3/2) \tag{11}$$

$$[ijkl] \frac{\partial}{\partial x^l} F_{jk} = 0 \tag{12}$$

$$\frac{\partial}{\partial x^k} \left[(-g)^{1/2} g^{ij} g^{kl} F_{jl} \right] = 0 \tag{13}$$

where the symbols have their usual meaning and ω is the coupling constant as understood in the BD theory.

Johri and Goswami have shown that corresponding to any diagonalizable solution of Einstein vacuum field equations in which fields and metric coefficients are functions of not more than three variables, we can generate a solution of the coupled B–D–Maxwell field equations with nonzero electromagnetic field. Mathematically, suppose the metric

$$ds^2 = e^{2W}(dx^3)^2 - e^{2W}\left[e_\alpha \Gamma\alpha\alpha (dx^\alpha)^2 \right] \tag{14}$$

with $\Gamma\alpha\alpha$ and W as functions of x^0, x^1, and x^2 satisfies Einstein's vacuum field equations; then the metric

$$ds^2 = e^{2\mu - EW/F}(dx^3)^2 - e^{-2\mu - EW/F}\left[e_\alpha \Gamma\alpha\alpha(dx^\alpha)^2\right] \tag{15}$$

will satisfy the BD–Maxwell field equations with scalar field given by

$$\phi = e^{EW/F} \tag{16}$$

E, F arbitrary constants given by

$$\frac{E}{F} = \pm\frac{2}{(3+2\omega)^{1/2}}$$

and related to potential c by

$$c + L = e^\mu \tag{17}$$

where L is a constant. The potential c may be obtained by

$$c = k\frac{e^{2W}-1}{e^{2W}+1}, \qquad k = \text{const} \tag{18}$$

The electromagnetic field is defined by

$$A = c\cos D, \qquad B = c\sin D, \qquad D = \text{const} \tag{19}$$

where A is the electric potential and B is the magnetic potential. When the fields are independent of x^0 instead of x^3, B would be the electric potential and A the magnetic potential.

4. BD–MAXWELL SOLUTIONS IN OBLATE SPHEROIDAL COORDINATES

Owing to the nonlinearity of the field equations one of the obstacles to a better understanding of the physical implications of general relativity is the relative scarcity of exact solutions. Hence generation of any new solution is always a useful step forward. We shall now use Johri and Goswami's method to generate a static electrovac B–D solution using Jipoy's solution in oblate spheroidal coordinates.

Using the line element (3) we get

$$g_{00}=\exp(2\mu-E\sigma/F)$$

$$=\left(k\frac{\exp(-2\beta\tan^{-1}\operatorname{csch}u)-1}{\exp(-2\beta\tan^{-1}\operatorname{csch}u)+1}+L\right)^2\exp\left(\frac{E}{F}\beta\tan^{-1}\operatorname{csch}u\right) \quad (20)$$

$$g_{33}=-\left[k\frac{\exp(-2\beta\tan^{-1}\operatorname{csch}u)-1}{\exp(-2\beta\tan^{-1}\operatorname{csch}u)+1}+L\right]^{-2}$$

$$\times a^2\cosh^2 u\cos^2\theta\exp\left(\frac{E}{F}\beta\tan^{-1}\operatorname{csch}u\right) \quad (21)$$

$$g_{11}=g_{22}=-\left[k\frac{\exp(-2\beta\tan^{-1}\operatorname{csch}u)-1}{\exp(-2\beta\tan^{-1}\operatorname{csch}u)+1}+L\right]^{-2}$$

$$\times\frac{a^2(\sinh^2 u+\sin^2\theta)^{\beta^2+1}}{(\cosh^2 u)^{\beta^2}}\exp\left(\frac{E}{F}\beta\tan^{-1}\operatorname{csch}u\right) \quad (22)$$

$$\phi=\exp\left(\frac{E\sigma}{F}\right)=\exp[-(E/F)\beta\tan^{-1}\operatorname{csch}u] \quad (23)$$

The electric potential

$$B=c\sin D=k\frac{\exp(-2\beta\tan^{-1}\operatorname{csch}u)-1}{\exp(-2\beta\tan^{-1}\operatorname{csch}u)+1}\sin D \quad (24)$$

In the asymptotic region where $u\to\infty$ our metric becomes flat, when L is put equal to unity and the electric field vanishes and the function ϕ becomes constant as expected.

Moreover, when $\omega\to\infty$, $\quad E/F\to 0$

Hence our metric reduces to the Einstein–Maxwell solution. This is also consistent with the requirement of the Brans–Dicke theory. This solution, however, is not exactly similar to the one we get using Harrison's method (Harrison, 1965) of generating the electromagnetic analog of Einstein's solution.

Let us study the asymptotic behavior of the above solution. It is evident from the transformation equation (2) that the oblate spheroidal coordinate θ

is measured from the equatorial plane. Now spherical polar coordinates $(r, \bar{\theta})$ are connected with cylindrical coordinates (ρ, z) by means of the equations

$$r = (\rho^2 + z^2)^{1/2}, \qquad \bar{\theta} = \sin^{-1} z/r \tag{25}$$

Contrary to the usual convention θ is here measured from the equator rather than from the pole for comparison with oblate spheroidal coordinates. Using equation (2) we obtain

$$r = a(\sinh^2 u + \cos^2 \theta)^{1/2} \underset{u \to \infty}{\to} a \sinh u \tag{26}$$

$$\bar{\theta} = \sin^{-1} \frac{a \sinh u \sin \theta}{r} \underset{u \to \infty}{\to} \theta \tag{27}$$

Using the above relations we get from the asymptotic expansions of the equations (20) and (24) the effective mass and the electric charge of the source to be $\beta a[k \pm 1/(3+2\omega)^{1/2}]$ and $\beta ak \sin D$, respectively. The second term within the bracket is evidently the contribution due to the B–D field. Moreover, the charge/mass ratio is a constant, as desired.

It is well known that in our coordinate system u-constant surfaces are oblate spheroid. Further, θ-constant surfaces are discontinuous as they cross the disc $u=0$, $\rho \leqslant a$. We can easily see that the derivatives $g_{\mu\nu,z}$ are discontinuous across the disc but the $g_{\mu\nu}$ themselves are continuous. Zipoy endowed the discontinuity with very complicated topologies. But Bonnor and Sackfield (1968) interpreted the discontinuity in the derivative as being due to the presence of a monopole layer of matter with Euclidean topology.

5. B–D–MAXWELL SOLUTIONS IN PROLATE SPHEROIDAL COORDINATES

Using equations (8) and (9) we now generate solutions in prolate spheroidal coordinates:

$$g_{00} = \left\{ k \frac{[(\cosh u + 1)/(\cosh u - 1)]^{-\beta} - 1}{[(\cosh u + 1)/(\cosh u - 1)]^{-\beta} + 1} + L \right\}^2 \left(\frac{\cosh u + 1}{\cosh u - 1} \right)^{-\beta/2(E/F)} \tag{28}$$

$$\phi=\left(\frac{\cosh u+1}{\cosh u-1}\right)^{\beta/2(E/F)} \tag{29}$$

$$B=k\frac{[(\cosh u+1)/(\cosh u-1)]^{-\beta}-1}{[(\cosh u+1)/(\cosh u-1)]^{-\beta}+1}\sin D \tag{30}$$

These solutions are also Enclidean at infinity and the scalar potential becomes constant and the electric field vanishes there. The charge/mass ratio once again is constant.

There is, however, a noted difference. When $u\to 0$, at the axis of symmetry the metric becomes singular. This is the case with Zipoy's solution also. There is thus one-to-one correspondence with Einstein's theory so far as singularity is concerned. So there exists a region of infinite red shift on the symmetry axis $\rho=0$, $-a\leqslant z\leqslant -a$. But unlike the case of oblate spheroid the θ-constant surfaces are no longer discontinuous across the $u=0$ disc.

6. RED SHIFT

Following the discovery of quasisteller objects with large red shifts, the question of gravitational red shifts has of late assumed an added significance (Burbidge and Burbidge, 1967). The red shifts were assumed to be cosmological. This assumption requires colossal amounts of energy to be released by the QSOs. Later observations showed that the case for noncosmological red shifts is strong (Burbidge, 1973; Sanders, 1974). It may not be quite out of place here to see whether our space-times also admit of large gravitational red shifts.

Bonnor and Wickramasuriya (1977) obtained an interior solution of Einstein–Maxwell equations for an oblate spheroid. We assume that our solution can also in principle be matched to an interior spheroid.

Let a photon in that case emanate from a point P on the surface of the spheroid and be received by an observer at a large distance where the space-time may be assumed to be Euclidean. Then the gravitational red shift is given by

$$\mathcal{G}=e^{-\sigma_P}-1 \tag{31}$$

$$=\exp\frac{\beta\tan^{-1}\operatorname{csch}u_0}{(3+2\omega)^{1/2}}\left[k\frac{\exp(-2\beta\tan^{-1}\operatorname{csch}u_0)-1}{\exp(-2\beta\tan^{-1}\operatorname{csch}u_0)+1}+1\right]^{-1}-1 \tag{32}$$

where $u=u_0$ characterizes the surface of the spheroid.

If $\mathcal{G}_{EM}$ refers to the red shift for the case of Einstein–Maxwell field and $\mathcal{G}_{BDM}$ refers to the case of BD–Maxwell field, they are connected by

$$\mathcal{G}_{BDM} = \mathcal{G}_{EM} \exp\left[\frac{\beta \tan^{-1} \operatorname{csch} u_0}{(3+2\omega)^{1/2}}\right] \qquad \text{(when the red shift is large).} \qquad (33)$$

Thus in the presence of the scalar field the red shift increases.

We may now consider two limiting cases keeping the mass constant:

$$\text{(a)} \quad u_0 \to 0, \qquad \mathcal{G} \to \infty$$

$$\text{(b)} \quad a \to 0, \qquad \mathcal{G} \to \infty$$

Thus in both cases the red shift becomes infinity. While in the first case the spheroid tends to a thin circular plate, in the second case it remains a spheroid but its volume tends to zero (a being a measure of the dimension of the spheroid). The cases with prolate spheroid are also more or less similar.

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